Published by Institute of Physics Publishing for SISSA
Received: *June 9, 2006*
Accepted: *October 3, 2006*
Published: *October 20, 2006*

Quantum mechanics of the doubled torus

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Abstract: We investigate the quantum mechanics of the doubled torus system, introduced by Hull [1] to describe T-folds in a more geometric way. Classically, this system consists of a world-sheet Lagrangian together with some constraints, which reduce the number of degrees of freedom to the correct physical number. We consider this system from the point of view of constrained Hamiltonian dynamics. In this case the constraints are second class, and we can quantize on the constrained surface using Dirac brackets. We perform the quantization for a simple T-fold background and compare to results for the conventional non-doubled torus system. Finally, we formulate a consistent supersymmetric version of the doubled torus system, including supersymmetric constraints.

Keywords: String Duality, Conformal Field Models in String Theory.

Contents

1. Introduction

In recent years there has been considerable interest in defining string theory on various "duality-folds" [1 − 15]. These backgrounds differ from ordinary manifolds because field configurations on overlapping coordinate patches can be glued together using duality transformations, as well as the conventional diffeomorphisms and gauge transformations. Of particular interest are backgrounds involving the T-duality group of transformations; these are known as "T-folds" [1]. These backgrounds are n-torus fibrations over some base, where the fibre undergoes monodromy transformations in the T-duality group $O(n, n; \mathbb{Z})$ around certain cycles in the base. T-folds are therefore fibre bundles with structure group $O(n, n; \mathbb{Z})$. A key feature of T-folds is that, unlike manifolds, they do not possess a globally well-defined metric. This is because T-duality transformations mix up the metric and B-field components. Nevertheless, sensible supergravity compactifications (typically using the Scherk-Schwarz ansatze [16, 17]) can be defined on these backgrounds [4, 6, 8 − 10, 18, 19].

In this paper we will be interested in the world-sheet description of T-folds. In particular, we will use a framework introduced by Hull known as the "doubled torus" formalism [1]. Essentially, the idea is to double the dimension of the T^n fibre, and consider the T-fold as a $2n$-dimensional torus fibration over the same base. Such ideas have also been implemented in earlier works, for example in refs. [20, 21]. The extra n dimensions are associated to the T-dual coordinates, $\tilde{X} = X_L - X_R$. By enlarging the fibre in this way, the monodromy transformations in $O(n, n; \mathbb{Z})$ act linearly. Moreover, since $O(n, n; \mathbb{Z})$ is a subgroup of $Gl(2n, \mathbb{Z})$, which is the group of large diffeomorphisms of T^{2n}, T-folds are geometric backgrounds from the doubled torus perspective. Physically speaking, one can think of the doubled torus as the set of all possible T-duals of a given T-fold [1].

Now, since the dimension of the fibre has been doubled, one must impose constraints to halve the number of physical degrees of freedom in order to make contact with critical string theory. In ref. [1] covariant constraints with the right properties are introduced. Therefore, the doubled torus model consists of a world-sheet Lagrangian, together with some constraints. These constraints can be imposed in a number of ways. One way is to solve the constraints and re-write everything in terms of the physical degrees of freedom. Using this approach Hull [1] shows that *classically* this leads to the conventional non-doubled formulation. In particular, by solving the constraints and using them in the doubled torus equations of motion, one arrives at equations of motion for a sigma model on the non-doubled torus, T^n. Furthermore, by choosing different "polarizations" for the physical coordinates one can obtain sigma models related to the original one by T-duality. In particular, the Buscher rules for the transformation of the metric and B-field under T-duality can be recovered.

Our approach will be to investigate the doubled torus formalism as a constrained Hamiltonian system. In particular, we will not solve the constraints, but rather we will impose them on our Hamiltonian, and then move to Dirac brackets in order to quantize on the constrained surface. The first aim of this paper is to investigate whether the doubled torus system is equivalent quantum mechanically to the more conventional non-doubled torus.

One hope is that the doubled torus formalism might be somewhat simpler quantum mechanically than the non-doubled torus. In the conventional formalism it is well known that understanding T-folds quantum mechanically involves the study of asymmetric orbifolds, which are non-trivial (see for example, refs. [11, 15, 22, 23]). However, since T-folds are geometric backgrounds from the doubled torus perspective, one might expect that there are no asymmetric orbifolds to deal with. This does not turn out to be the case. In fact, we recover exactly the same (asymmetric) orbifolds and partition functions as in the conventional case, even though the steps along the way are somewhat different.

The second aim of this paper is to find the supersymmetric version of the doubled torus model. Although there is much work on supersymmetric sigma models [24 − 28], in this case one has to also consider how to make the constraints supersymmetric. We will construct a consistent supersymmetric Lagrangian with suitable constraints.

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The plan of this paper is as follows. Firstly, in section 2 we review the work of ref. [1]. Then in section 2.2 we move to the Hamiltonian formulation and determine the class of constraints we are dealing with. After establishing that the constraints are second class we move to Dirac brackets. Since the Dirac brackets are very simple we are able to quantize canonically, without invoking BRST quantization (which would involve making our constraints first class), or other more complicated methods. To actually perform the quantization we consider a very simple T-fold in section 3 from the doubled perspective. Our quantization takes place on the constrained surface, which is a surface in phase space. An attractive feature of our analysis is that it does not require a choice of polarization to be made. We calculate all the quantum mechanical ingredients such as Virasoro operators, the Hilbert space, the partition function and so on, and compare to the non-doubled results. In section 4 we give our results for a supersymmetric version of the doubled torus formalism, including the supersymmetrized constraint. In section 5 we discuss our results and conclude.

Note added: After our paper first appeared on the archive, a new paper by C. Hull [29] appeared which discusses important aspects of the doubled torus formalism. These include the quantum equivalence to the usual formulation, arbitrary genus worldsheets and the dilaton. The method of quantization involves gauging half of the currents, and is different to the method used here.

2. Bosonic theory and constraint analysis

We begin by considering the doubled torus system defined by Hull in ref. [1]. This is a constrained Lagrangian system, where the degrees of freedom on the fibre are doubled; constraints are then imposed to reduce these degrees of freedom to the correct physical number. We will analyse this setup as a constrained *Hamiltonian* system. This will lead to a natural quantization in terms of Dirac brackets.

2.1 Review of doubled torus formulation

In this section we review the doubled torus construction for T-folds. The starting point is to consider a sigma model, defined by embedding coordinates $(X^I,\, Y^m)$ which map the world-sheet into the target space. Locally, the target space takes the form $N \times T^{2n}$, where T^{2n} is the doubled torus. Globally, however, the target space is a T^{2n} fibre bundle over N, with structure group $O(n,n;\mathbb{Z})$. The embedding coordinates X^I are associated to T^{2n}, hence we have the periodicity conditions[1] $X^I \sim X^I + 2\pi$, where $I = 1, \ldots, 2n$. The coordinates Y^m are associated to the base, so $m = 0, 1, \ldots, 26 - n$. Our total number of dimensions is $26 + n$, but n of these will be unphysical.

The data on the target space consists of a generalised metric, H_{IJ}, and source terms, J_I, on T^{2n}, together with a metric, G_{mn}, and B-field, B_{mn}, on the base. The following Lagrangian can be constructed for this system [1],

$$\mathcal{L} = \frac{1}{2} H_{IJ}(Y) P^I \wedge \star P^J + P^I \wedge \star J_I(Y) + \mathcal{L}(Y) \tag{2.1}$$

[1] This does not mean that all radii in the T^{2n} are equal, but rather the radii will enter in the metric H_{IJ}

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where $P^I = dX^I$, and d, $\wedge$ and $\star$ are all operations on the worldsheet. The Lagrangian on the base space, $\mathcal{L}(Y)$, can be taken to have the following general form,

$$\mathcal{L}(Y) = \frac{1}{2} G_{mn} dY^m \wedge \star dY^n + \frac{1}{2} B_{mn} dY^m \wedge dY^n$$

while the source terms can be expressed in the following natural way, $J_I = A_{In} dY^n + \tilde{A}_{In} \star dY^n$. Then the Lagrangian is

$$\begin{aligned}\mathcal{L} = & \frac{1}{2} H_{IJ} \partial_a X^I \partial_b X^J \eta^{ab} + A_{In} \partial_a X^I \partial_b Y^n \eta^{ab} - \tilde{A}_{In} \partial_a X^I \partial_b Y^n \epsilon^{ab} \\ & + \frac{1}{2} G_{mn} \partial_a Y^m \partial_b Y^n \eta^{ab} - \frac{1}{2} B_{mn} \partial_a Y^m \partial_b Y^n \epsilon^{ab} \end{aligned} \tag{2.2}$$

where $\sigma^{a,b} = \tau, \sigma$ are the world-sheet coordinates, $\eta = \mathrm{diag}(+1, -1)$ is the flat world-sheet metric and $\epsilon_{01} = +1$. All fields in the above Lagrangian are assumed to depend on the base coordinates, Y^m, in general.

By varying X^I one obtains the following equation of motion,

$$d \star (HP + J) = 0 \tag{2.3}$$

This can be written more explicitly as

$$\eta^{ab} \partial_a (H_{IJ} \partial_b X^J + A_{In} \partial_b Y^n) - \epsilon^{ab} \partial_m \tilde{A}_{In} \partial_a Y^m \partial_b Y^n = 0$$

Physical solutions of this equation should also satisfy the following constraint (which is really n constraints), which halves the number of physical degrees of freedom

$$\star P^I = S^I{}_J P^J + L^{IJ} J_J \tag{2.4}$$

where L^{IJ} is a constant $O(n,n)$ invariant metric[2] and $S^I{}_J \equiv L^{IK} H_{KJ}$. In the next section we will see precisely why this constraint halves the number of degrees of freedom. First, however, we see that for the consistency of the constraint one must have $S^2 = 1$ and $SL \star J = -LJ$. This restricts the form of H_{IJ} and implies

$$A_{In} = -H_{IJ} L^{JK} \tilde{A}_{Kn} \tag{2.5}$$

for the constituents $A, \tilde{A}$ of the source term J_I.

An important feature of the doubled torus system is that it is invariant under $O(n,n;\mathbb{Z})$. Suppose we consider a global $O(n,n)$ transformation, M (which must satisfy $M^T L M = L$), then X, H, J transform as follows,

$$\begin{aligned} X &\to MX \\ H &\to (M^{-1})^T H M^{-1} \\ J &\to (M^{-1})^T J \end{aligned} \tag{2.6}$$

[2] $O(n,n)$ matrices M must therefore satisfy $M^T L M = L$.

Hence the Lagrangian (2.1) and constraint (2.4) remain invariant under the continuous group $O(n,n)$. However, only the discrete subgroup $O(n,n;\mathbb{Z})$ will leave the lattice for X^I invariant.

To make contact with the conventional formulation of bosonic strings on T^n, one must divide the coordinates on T^{2n} into n physical coordinates, $X^i \in T^n$, and n dual coordinates, $\tilde{X}_i \in \tilde{T}^n$. This is referred to as a "choice of polarization". In group theoretic language, we are decomposing $O(n,n)$ into representations of $GL(n)$. In particular, the $2n$-dimensional representation of $O(n,n)$ decomposes as $2n \to n + n'$, where n and n' are the fundamental and anti-fundamental representations of $GL(n)$. This decomposition can be implemented in a geometric way by the following $2n \times 2n$ matrix [1],

$$\Pi = \begin{pmatrix} \Pi^i{}_I \\ \tilde{\Pi}_{iI} \end{pmatrix}$$

where upper i indices correspond to the n representation, and lower i indices correspond to n'. The physical subspace T^n must be a null subspace with respect to the constant $O(n,n)$ metric L, i.e.

$$\Pi^i{}_I \Pi^j{}_J L^{IJ} = \tilde{\Pi}_{iI} \tilde{\Pi}_{jJ} L^{IJ} = 0$$

Also,

$$\Pi^i{}_I \tilde{\Pi}_{iJ} + \tilde{\Pi}_{iI} \Pi^i{}_J = L_{IJ}$$

In terms of the $GL(n)$ basis, L_{IJ} can always be taken to be

$$L_{IJ} = \begin{pmatrix} 0 & 1_{n\times n} \\ 1_{n\times n} & 0 \end{pmatrix} \tag{2.7}$$

which gives the natural metric $ds_L^2 = 2dX^i d\tilde{X}_i$. Moreover, in a certain gauge [1] the metric H_{IJ} can be chosen to take the conventional form for a $O(n,n)/O(n) \times O(n)$ coset metric, namely

$$H = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix} \tag{2.8}$$

where G and B are the metric and B-field on T^n. Notice that if we take L and H as above then $S^2 = 1$ automatically, and also $\mathrm{Tr}S = 0$.

The distinguishing feature of T-folds, compared to ordinary manifolds, is that in general no global polarization, Π, can be chosen, even though it is always possible locally. This is equivalent to the earlier statement about T-folds, namely that they have no globally well defined metric. Hence the above form for H_{IJ} only makes sense locally.

At this point we take a different approach to ref. [1]. Instead of solving the constraint (2.4) for $\tilde{X}$ in terms of the other quantities, we will consider the doubled torus as a constrained Hamiltonian system. In particular, we will determine the class of the constraint we have here, and then use methods from constrained Hamiltonian dynamics to quantize on the constrained surface.

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2.2 Hamiltonian and constraint analysis

We can write the total Lagrangian (2.2) in a more compact form as

$$\mathcal{L} = \frac{1}{2} g_{\mu\nu} \dot{q}^\mu \dot{q}^\nu + \dot{q}^\mu j_\mu - V[q] \tag{2.9}$$

where the indices $\mu = I, n$, we define $q^I = X^I$, $q^n = Y^n$ and $\dot{q}^\mu \equiv \partial_\tau q^\mu$. The metric is

$$g_{\mu\nu} = \begin{pmatrix} H_{IJ} & A_{In} \\ A_{Jm} & G_{mn} \end{pmatrix} \tag{2.10}$$

The source terms j_μ are given by $j_I = \tilde{A}_{In} Y'^n$, $j_n = -\tilde{A}_{In} X'^I + B_{nm} Y'^m$, and the potential is $V[q] = \frac{1}{2} g_{\mu\nu} q'^\mu q'^\nu$, where $q'^\mu \equiv \partial_\sigma q^\mu$.

The conjugate momenta are $\pi_\mu = g_{\mu\nu}\dot{q}^\nu + j_\mu$. More explicitly, the conjugate momentum of X^I is

$$\pi_I = \frac{\partial L}{\partial \dot{X}^I} = H_{IJ} \dot{X}^J + A_{In} \dot{Y}^n + \tilde{A}_{In} Y'^n$$

and the conjugate momenta of Y^n is

$$\pi_n = \frac{\partial L}{\partial \dot{Y}^n} = G_{mn} \dot{Y}^m + A_{In} \dot{X}^I - \tilde{A}_{In} X'^I + B_{nm} Y'^m$$

This allows us to calculate the Hamiltonian density, $\mathcal{H}$. We find,

$$\begin{aligned} \mathcal{H} &= \pi_\mu \dot{q}^\mu - \mathcal{L} \\ &= \frac{1}{2} g^{\mu\nu} (\pi_\mu - j_\mu)(\pi_\nu - j_\nu) + \frac{1}{2} g_{\mu\nu} q'^\mu q'^\nu \end{aligned} \tag{2.11}$$

Here we see that the Hamiltonian is only well-defined if $g_{\mu\nu}$ is invertible. Note that H_{IJ} and G_{mn} being invertible do not guarantee that $g^{\mu\nu}$ exists. However, for our analysis we will require $g_{\mu\nu}$ to be invertible.

We now discuss the constraint which we want to impose on this Hamiltonian system. Recall that the constraint (2.4) is

$$\star P^I = S^I{}_J P^J + L^{IJ} J_J$$

Writing it in its two components gives

$$\Phi_1^- = P_\tau - S P_\sigma - L J_\sigma = 0$$

$$\Phi_2^- = P_\sigma - S P_\tau - L J_\tau = 0$$

where we are omitting I, J indices for brevity. Taking sums and differences of these two equations, one finds

$$\frac{1}{2}(1 \pm S)(P_\tau \mp P_\sigma) = \frac{1}{2} L (J_\sigma \mp J_\tau)$$

Now since $S^2 = 1$ and $\mathrm{Tr} S = 0$, this means that $(1 \pm S)/2$ are projectors onto two orthogonal n-dimensional subspaces. Therefore, the constraint is forcing half of the X^Is to be purely left moving, and half to be purely right moving.

In fact, using $S^2 = 1$ and $SL\star J = -LJ$, one finds $\Phi_2^- = -S\Phi_1^-$ and so we can take Φ_1^- as our only primary constraint. Using the consistency conditions one finds we can rewrite Φ_1^- as follows,

$$\Phi_1^{-I} = H^{IJ}(\pi_J - L_{JK}P_\sigma^K)$$

where π_J is the conjugate momentum for X^J defined above. Therefore, our primary constraint can be taken to be in the form

$$\Phi_I^- \equiv \pi_I - L_{IJ}X'^J \tag{2.12}$$

We now calculate the Poisson bracket of the constraint Φ^- with various other quantities. This will allow us to determine whether there are secondary constraints and the Dirac class of the constraints. Recall that the canonical Poisson brackets are

$$\left\{X^I(\sigma), \pi_J(\sigma')\right\}_{\rm PB} = \delta_J^I\delta(\sigma - \sigma')$$

$$\left\{Y^n(\sigma), \pi_m(\sigma')\right\}_{\rm PB} = \delta_m^n\delta(\sigma - \sigma')$$

We consider the time evolution and closure of the constraint. We find,

$$\left\{\Phi_I^-(\sigma), \int_{\sigma'}\mathcal{H}\right\}_{\rm PB} = \partial_\sigma(-L_{IJ}H^{JK}\Phi_K^-) \simeq 0$$

$$\left\{\Phi_I^-(\sigma_1), \Phi_J^-(\sigma_2)\right\}_{\rm PB} = -2L_{IJ}\delta'(\sigma_1 - \sigma_2)$$

This means that there are no secondary constraints and our constraint, Φ_I^-, is second class. By imposing it we can safely reduce our theory on the constrained phase space $\Phi_I^- = 0$ leaving no other symmetry or gauge freedom. On that surface the dynamics are described by the Dirac bracket,

$$\{A, B\}_{\rm D} = \{A, B\}_{\rm PB} - \int_{\sigma,\sigma'}\left\{A, \Phi_I^-(\sigma)\right\}_{\rm PB} G^{IJ}(\sigma, \sigma')\left\{\Phi_J^-(\sigma'), B\right\}_{\rm PB}$$

where

$$G^{IJ}(\sigma, \sigma') = \left\{\Phi_I^-(\sigma), \Phi_J^-(\sigma')\right\}_{\rm PB}^{-1} = -\frac{1}{4}L^{IJ}\left(\epsilon(\sigma - \sigma') - \epsilon(\sigma' - \sigma)\right)$$

and ϵ is the Heaviside step function. We find the following Dirac brackets,

$$\left\{X^I(\sigma), X^J(\sigma')\right\}_{\rm D} = -\frac{1}{4}L^{IJ}\left(\epsilon(\sigma - \sigma') - \epsilon(\sigma' - \sigma)\right)$$

$$\left\{X^I(\sigma), \pi_J(\sigma')\right\}_{\rm D} = \frac{1}{2}\delta_J^I\delta(\sigma - \sigma')$$

$$\left\{\pi_I(\sigma), \pi_J(\sigma')\right\}_{\rm D} = \frac{1}{2}L_{IJ}\delta'(\sigma - \sigma')$$

Moreover, we can define the rotated coordinates $\Phi_I^+ = \pi_I + L_{IJ}X'^J$, and the Dirac brackets of $\Phi_I^\pm$ are given by

$$\left\{\Phi_I^-(\sigma), A\right\}_{\rm D} = 0$$

$$\left\{\Phi_I^-(\sigma), \Phi_J^+(\sigma')\right\}_{\rm D} = 0$$

$$\left\{\Phi_I^+(\sigma), \Phi_J^+(\sigma')\right\}_{\mathrm{D}} = 2L_{IJ}\delta'(\sigma - \sigma')$$

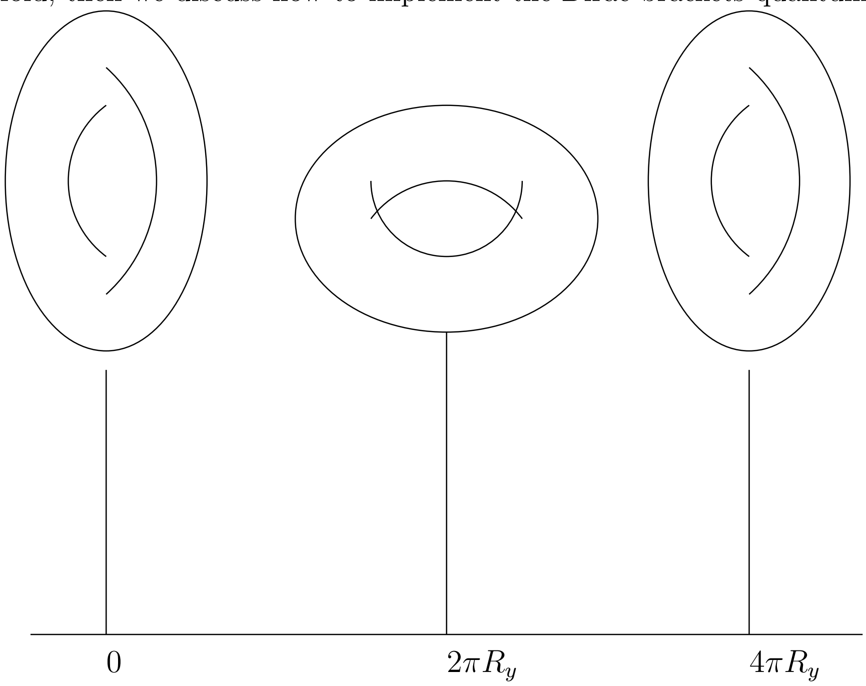
where A is any quantity. The coordinates Φ_I^+ can be thought of as tangent to the constraint surface, $\Phi_I^- = 0$. In terms of the coordinates $\Phi_I^\pm$, the Hamiltonian can be written as

$$\mathcal{H} = \frac{1}{2}g^{\mu\nu}Z_\mu Z_\nu - \frac{1}{4}H^{IJ}\Phi_I^+\Phi_J^+ + \frac{1}{4}H^{IJ}\Phi_I^-\Phi_J^- + \frac{1}{2}G_{mn}Y'^m Y'^n \tag{2.13}$$

with $Z_I = \Phi_I^+ - \tilde{A}_{In}Y'^n$, $Z_m = \pi_m - B_{mn}Y'^n$. Notice that Φ^- only appears quadratically. This will be important later.

3. Bosonic orbifold

In this section we quantize a simple example of a T-fold from the doubled, constrained Hamiltonian perspective. In the non-doubled language the T-fold we are interested in has a S^1 fibre over an S^1 base, with a T-duality acting on the fibre as one traverses the base. First, we describe the doubled description of this background, including the relevant orbifold; then we discuss how to implement the Dirac brackets quantum mechanically.



3.1 The setup

In the doubled language our T-fold corresponds to a T^2 fibred over S^1. We take the coordinates on the doubled fibre to be X^1, X^2, while Y will be the coordinate on the base S^1. As one traverses the base S^1 the fibre undergoes a monodromy transformation, where the monodromy is the only non-trivial element of $O(1,1;\mathbb{Z})$. From the non-doubled point of view, this corresponds to ordinary T-duality on an S^1 fibre, i.e. $R \to R^{-1}$. As we will see, the monodromy will act naturally on the doubled coordinates. Moreover, it constitutes a geometric transition function for the doubled torus.

So locally our background looks like

$$N \times S^1 \times T^2$$

where N is taken to be some flat manifold with coordinates Y^a. For simplicity, we turn all B fields off. That is, we set

$$A_{mI} = \tilde{A}_{mI} = B_{mn} = 0$$

and also require no Y dependence

$$\partial_n H_{IJ} = \partial_n G_{mn} = 0$$

where $Y^m = (Y^a, Y)$.

We now construct the orbifold using the following identifications.

$$\begin{aligned} X^I &\to M^I{}_J X^J \\ Y &\to Y + 2\pi R_y \\ Y^a &\to Y^a \end{aligned}$$

where Y is the coordinate on a circle with radius $2R_y$ (i.e. $Y \equiv Y + 4\pi R_y$), so this corresponds to a half shift around the circle. Therefore, our orbifold is of order 2. The associated transformation for the metric H_{IJ} is

$$H \to (M^{-1})^T H M^{-1}$$

Using the coset form for the metric H, we have

$$H = \begin{pmatrix} R^2 & 0 \\ 0 & R^{-2} \end{pmatrix}$$

where R is the radius of the original S^1 fibre (in the non-doubled picture). Here the monodromy matrix $M \in O(1,1;\mathbb{Z}) \subset GL(2;\mathbb{Z})$ will be the only non-trivial possibility, namely $M^I{}_J = L^{IJ}$:

$$M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3.1}$$

Therefore, using the transformation rule for H, we see that our monodromy corresponds to

$$R \to \frac{1}{R}$$

which is what we want.

3.2 Equations of motion

We now consider this particular doubled torus orbifold from the point of view of the constrained Hamiltonian system. Recall that we began with phase space (X^I, π_I, Y^n, π_n) but on the reduced surface $\Phi^- = 0$, so the phase space is (Φ^+_I, Y^n, π_n). Henceforth we put

$\Phi_I^- = 0$ and use the symbol Φ_I for Φ_I^+. From (2.13) the Hamiltonian is

$$\mathcal{H} = \frac{1}{2} g^{\mu\nu} Z_\mu Z_\nu - \frac{1}{4} H^{IJ} \Phi_I \Phi_J + \frac{1}{2} G_{mn} Y'^m Y'^n$$

with $Z_I = \Phi_I$, $Z_m = \pi_m$ here. The non-trivial Dirac brackets are

$$\begin{aligned} \left\{ Y^n(\sigma), \pi_m(\sigma') \right\}_{\mathrm{D}} &= \delta^n_m \delta(\sigma - \sigma') \\ \left\{ \Phi_I(\sigma), \Phi_J(\sigma') \right\}_{\mathrm{D}} &= 2 L_{IJ} \delta'(\sigma - \sigma') \end{aligned}$$

The equations of motion $\dot{f} = \left\{ f, \int_{\sigma'} H(\sigma') \right\}_{\mathrm{D}}$ are

$$\dot{\Phi}_I = L_{IJ} H^{JK} \Phi'_K \tag{3.2}$$

$$d \star dY = d \star dY^a = 0 \tag{3.3}$$

Note that the equation of motion for Φ_I is different from equations of motion one would obtain from the doubled torus Lagrangian. This is because we are now considering dynamics on the constrained surface. We can solve (3.2) by diagonalising $LH^{-1} = S^T$ into ± 1 eigenspaces.

$$S^T = LH^{-1} = \begin{pmatrix} 0 & R^2 \\ R^{-2} & 0 \end{pmatrix}$$

Then we obtain the following solution for Φ_I,

$$\Phi_I(\sigma, \tau) = \Phi_{0I} + \Phi_I^{(+1)}(\sigma^+) + \Phi_I^{(-1)}(\sigma^-)$$

where Φ_{0I} is constant, and $\Phi_I^{(\pm 1)}$ are ± 1 eigenvectors of S^T. The periodicities of $\Phi_I^{(\pm 1)}(\sigma^\pm)$ will be determined by the particular boundary conditions we choose. Note that Φ_I will not have any linear terms in $\sigma^\pm$. This is because $\Phi_I = \Pi_I + L_{IJ} X'^J$ and both Π_I and X'^J are periodic.

The solution for (3.3) is

$$\begin{aligned} Y =& Y_R(\sigma^-) + Y_L(\sigma^+) \\ Y_R(\sigma^-) =& \frac{1}{2} y_0 + p_R \sigma^- + \frac{1}{\sqrt{2}} \sum_{k \neq 0} \frac{i b_k}{k} e^{-ik\sigma^-} \\ Y_L(\sigma^+) =& \frac{1}{2} y_0 + p_L \sigma^+ + \frac{1}{\sqrt{2}} \sum_{k \neq 0} \frac{i \tilde{b}_k}{k} e^{-ik\sigma^+} \end{aligned} \tag{3.4}$$

In the next section we will use the boundary conditions for Y to determine a quantization rule for p_L and p_R. Similarly, solving (3.3) for the rest of the coordinates, Y^a, one obtains

$$Y^a = y_0^a + p^a \tau + \frac{1}{\sqrt{2}} \sum_{k \neq 0} \frac{i b_k^a}{k} e^{-ik\sigma^-} + \frac{1}{\sqrt{2}} \sum_{k \neq 0} \frac{i \tilde{b}_k^a}{k} e^{-ik\sigma^+} \tag{3.5}$$

For the orbifold we are interested in here, we can distinguish two twisted sectors. These sectors will give us the boundary conditions we need to fix the solutions Φ_I and Y completely. Sector I has

$$\Phi_I(\sigma + 2\pi) = \Phi_I(\sigma)$$

$$Y(\sigma+2\pi) = Y(\sigma) + 4\pi R_y m \qquad m\in\mathbb{Z}$$

Sector II has

$$\begin{aligned}\Phi_I(\sigma+2\pi) &= M_I{}^J\Phi_J(\sigma)\\ Y(\sigma+2\pi) &= Y(\sigma) + 2\pi R_y(2m+1) \qquad m\in\mathbb{Z}\end{aligned}$$

So our two sectors are distinguished by whether we shift an odd or even multiple of $2\pi R_y$ around the base circle. We will now consider each of these sectors in turn.

3.3 Sector I

We have the boundary conditions

$$\begin{aligned}\Phi_I(\sigma+2\pi) &= \Phi_I(\sigma)\\ Y(\sigma+2\pi) &= Y(\sigma) + 4\pi R_y m \qquad m\in\mathbb{Z}\end{aligned}$$

From these boundary conditions we see that $\Phi^{\pm}_I(\sigma^{\pm})$ are periodic functions. Therefore, the solution for Φ_I is

$$\Phi_I(\sigma,\tau) = \begin{pmatrix} q_1\\ q_2\end{pmatrix} + \begin{pmatrix} R\\ R^{-1}\end{pmatrix}\sum_{k\neq 0}\tilde{a}_k e^{-ik\sigma^+} + \begin{pmatrix} R\\ -R^{-1}\end{pmatrix}\sum_{k\neq 0} a_k e^{-ik\sigma^-} \tag{3.6}$$

where the vectors

$$e_{\pm}(R) \equiv \begin{pmatrix} R\\ \pm R^{-1}\end{pmatrix}$$

are the ± 1 eigenstates of S^T. The constants q_1, q_2 are related to the winding and momentum quantum numbers that would appear in the conventional non-doubled formalism. In appendix B we show that q_1, q_2 obey the quantization condition $q_1 q_2 = 2mn$, where $m, n\in\mathbb{Z}$.

We now turn to the boundary conditions for Y. The solution for Y is given in (3.4), and the boundary conditions give

$$\begin{aligned} p_L + p_R &\in \frac{1}{2R_y} n_y \qquad n_y\in\mathbb{Z}\\ p_L - p_R &\in 2R_y w_y \qquad w_y\in\mathbb{Z}\end{aligned}$$

Now we are ready to quantize Φ_I, Y^a, Y. We will begin with $Y^m = (Y^a, Y)$, $m = 0,\dots 24$ with $Y^{24}\equiv Y$. For our background we have $\pi_n = \eta_{nm}\dot{Y}^m$, and we want to impose the following bracket as an operator relation,

$$\{Y^m(\sigma,\tau), \pi_n(\sigma',\tau)\} = \delta^m_n\delta(\sigma-\sigma')$$

By replacing $\{,\}\to -i[,]$ we obtain the following commutation relations for the modes associated to Y^m,

$$[b^m_k, b^n_l] = k\delta_{k+l}\eta^{mn} \qquad [\tilde{b}^m_k, \tilde{b}^n_l] = k\delta_{k+l}\eta^{mn} \qquad [b^m_k, \tilde{b}^n_l] = 0 \tag{3.7}$$

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Now we consider Φ_I. We have the following Dirac brackets,

$$\left\{\Phi_I(\sigma,\tau),\Phi_J(\sigma',\tau)\right\}_{\mathrm{D}} = 2L_{IJ}\delta'(\sigma-\sigma')$$

Replacing $\{,\}_{\mathrm{D}} \to -i[,]$ one arrives at the following

$$[a_k, a_l] = k\delta_{k+l} \qquad [\tilde{a}_k, \tilde{a}_l] = k\delta_{k+l} \qquad [a_k, \tilde{a}_l] = 0$$

The Hilbert space for this sector, denoted by $H_{(+)}$, will be built on a vacuum $|0>$ that is invariant under the monodromy, i.e. $M|0>=|0>$. The states we construct will be "off-shell" since we haven't yet imposed physical state conditions. We decompose the Hilbert space for this sector into eigenspaces $H^{\pm}_{(+)}$ associated to eigenvalues ±1 under the orbifold action, M. Under M we have $\Phi_I \to M_I{}^J\Phi_J$, so using the explicit form for M given in (3.1) this corresponds to the following action on the eigenvectors which appear in the decomposition (3.6):

$$e_{\pm}(R) \longrightarrow \pm e_{\pm}(R^{-1})$$

together with $q_1 \leftrightarrow q_2$. Therefore, the associated action on the modes is

$$\tilde{a}_k \mapsto \tilde{a}_k, \qquad a_k \mapsto -a_k$$

As explained in the recent paper ref. [15], the correct action of T-duality on states involves a non-trivial phase. This can be shown by considering the OPE of two $+1$ T-eigenstates and requiring that no -1 eigenstates appear on the right hand side [15]. In our doubled language the correct action of T-duality is

$$T|q_1,q_2> = (-1)^{\frac{q_1q_2}{2}}|q_2,q_1> \tag{3.8}$$

This phase is essential for modular invariance of the resulting partition function, as we will see. Hence the Hilbert space for the non-trivial fibre bundle part of the space-time splits up into $H_{(+)} = H^+_{(+)} \oplus H^-_{(+)}$, where

$$H^{\pm}_{(+)} = \left\{\prod_{i=1}^{N} a_{-n_i}\prod_{j,k,l}\tilde{a}_{-m_j}b_{-r_k}\tilde{b}_{-s_l}\left(|q_1,q_2;n_y,w_y> \pm(-1)^{N+n_y+\frac{q_1q_2}{2}}|q_2,q_1;n_y,w_y>\right)\right\}$$

Note that the factor of $(-1)^{n_y}$ is due to the Y-shift in our orbifold. See appendix B for arguments which lead to the quantization rule $q_1q_2 = 2mn$.

Before we move on to sector II we point out a few interesting features. Firstly, we only have one set of left-moving modes and one set of right-moving modes from $\Phi_I(\sigma,\tau)$. This means that there is no need to make a choice of polarization for our quantum mechanical states. This is in contrast to the Lagrangian formulation [1], where classically a polarization must be chosen to make contact with the non-doubled formulation. Here we do not need to choose polarization because we have moved to the constrained surface in phase space.

Secondly, we notice that our orbifold looks very similar to the interpolating orbifolds considered in [11, 15]. This suggests that the doubled torus formalism is equivalent to the conventional non-doubled formulation of these backgrounds. However, we must work out the precise details since we are quantizing Φ, not X, and so there may be differences in, for example, the physical state conditions or the partition function.

3.4 Sector II

In this sector we have the boundary conditions

$$\begin{aligned}\Phi_I(\sigma+2\pi) &= M_I{}^J\Phi_J(\sigma)\\ Y(\sigma+2\pi) &= Y(\sigma)+(2m+1)2\pi R_y \qquad\qquad m\in\mathbb{Z}\end{aligned}$$

The solutions for Y and Y^a are unchanged from sector I and are given in (3.4)-(3.5). However, the quantization conditions for p_L, p_R are now

$$\begin{aligned}p_L+p_R &\in \frac{n_y}{2R_y} \qquad\qquad n_y\in\mathbb{Z}\\ p_L-p_R &\in R_y(2w_y+1) \qquad\qquad w_y\in\mathbb{Z}\end{aligned}$$

due to the new boundary conditions on Y. The oscillator algebras for the modes associated to Y and Y^a are unchanged from sector I and are given in (3.7). We now turn our attention to Φ_I. The solution for Φ_I can still be written as

$$\Phi_I = \Phi_{0I} + e_+(R)f(\sigma^+) + e_-(R)g(\sigma^-)$$

but now the boundary conditions imply that f is periodic while g is anti-periodic. Therefore, Φ_I can be expanded in modes as

$$\Phi_I(\sigma,\tau) = \begin{pmatrix} q \\ q \end{pmatrix} + e_+(R)\sum_{k\neq 0}\tilde{a}_k e^{-ik\sigma^+} + e_-(R)\sum_{k\in\mathbb{Z}+\frac{1}{2}} a_k e^{-ik\sigma^-}$$

where now the boundary conditions for this sector force the constant term $\Phi_{01}=\Phi_{02}=q$. The quantization condition on q is

$$q = \frac{1}{\sqrt{2}}\left(n-\frac{1}{2}\right), \qquad n\in\mathbb{Z} \tag{3.9}$$

This condition is chosen so that level matching in this sector makes sense [15] (see next section). We will prove this is the correct quantization in appendix B.

Again we want to impose the following (Dirac) bracket as an operator relation for the modes,

$$[\Phi_I(\sigma,\tau),\Phi_J(\sigma',\tau)] = 2iL_{IJ}\delta'(\sigma-\sigma')$$

Note that $\delta'(\sigma-\sigma')$ cannot simply be periodic since this will not be compatible with the monodromy transformations as $\sigma\to\sigma+2\pi$. To get a correct global statement we should replace the right hand side of the bracket with the monodromy invariant

$$2iL_{IJ}\delta'(\sigma-\sigma') = i\begin{pmatrix} R^2\left[\delta'_{2\pi}(\Delta\sigma)-\delta'_{4\pi}(\Delta\sigma)\right] & \delta'_{2\pi}(\Delta\sigma)+\delta'_{4\pi}(\Delta\sigma) \\ \delta'_{2\pi}(\Delta\sigma)+\delta'_{4\pi}(\Delta\sigma) & R^{-2}\left[\delta'_{2\pi}(\Delta\sigma)-\delta'_{4\pi}(\Delta\sigma)\right]\end{pmatrix}$$

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where $\Delta\sigma \equiv \sigma - \sigma'$ and $\delta_{2\pi}, \delta_{4\pi}$ are delta functions with period 2π and 4π respectively. Then the commutation relations for the modes are

$$[a_k, a_l] = k\delta_{k+l} \qquad [\tilde{a}_m, \tilde{a}_n] = m\delta_{m+n} \qquad [a_k, \tilde{a}_m] = 0 \tag{3.10}$$

where $k, l \in \mathbb{Z} + \frac{1}{2}$ and $m, n \in \mathbb{Z}$.

We now discuss the (off-shell) Hilbert space of sector II, which will be denoted by $H_{(-)}$. First note that we need a twisted vacuum for the right-handed module so that the vacuum flips sign under the monodromy,[3] i.e. $M|0>_- = -|0>_-$. As in sector I the action of the monodromy on the modes is

$$\tilde{a}_k \mapsto \tilde{a}_k, \qquad a_k \mapsto -a_k$$

In this sector there is also a non-trivial phase to take into account, namely

$$T|q> = e^{-\frac{i\pi}{8}}(-1)^{q^2}|q> \tag{3.11}$$

This phase has been proved (in the non-doubled formulation) by Hellerman and Walcher [15] using OPE relations.

So we have the decomposition of the Hilbert space $H_{(-)} = H^+_{(-)} \oplus H^-_{(-)}$, into ± 1 eigenstates under the monodromy, where

$$H^{\pm}_{(-)} = \left\{ \frac{1 \pm (-1)^{N+n_y+n(n-1)/2}}{2} \prod_{i=1}^{N} a_{-n_i} \prod_{j,k,l} \tilde{a}_{-m_j} b_{-r_k} \tilde{b}_{-s_l} |q, n_y, w_y>_- \right\}$$

and $n \in \mathbb{Z}$ is related to q by (3.9). Here the factor of $(-1)^{n_y}$ comes from the Y-shift, as before.

3.5 Physical state conditions

In this section we consider the physical state conditions for the eigenstates we have constructed above for $H_{(\pm)}$. In particular, we will investigate the level matching conditions, mass formulae and ultimately the partition function for this particular doubled torus setup. Our goal is to show that quantizing the doubled torus using the constrained Hamiltonian systems method is equivalent to quantizing the non-doubled torus.

To begin, we will calculate the energy-momentum tensor from the doubled torus Lagrangian (2.9). As usual, this is defined as

$$T_{ab} = \frac{2}{\sqrt{-h}} \frac{\partial \mathcal{L}}{\partial h^{ab}}\Big|_{h=\eta}$$

where h_{ab} is a general world-sheet metric. One finds,

$$T_{ab} = g_{\mu\nu}\partial_a q^\mu \partial_b q^\nu - \frac{1}{2}\eta_{ab}\eta^{cd} g_{\mu\nu}\partial_c q^\mu \partial_d q^\nu \tag{3.12}$$

Due to Weyl invariance $T_{00} = T_{11}$, so we need only investigate T_{00} and T_{01}. Written in phase space, for generic $g_{\mu\nu}, j_\mu$, one finds

$$T_{00} = \mathcal{H}$$

[3] This is evident in radial quantization and CFT for states to be well-defined.

$$T_{01} = \pi_\mu q^{\mu\prime} = \frac{1}{4}L^{IJ}\Phi_I^+\Phi_J^+ + \pi_m Y^{m\prime} - \frac{1}{4}L^{IJ}\Phi_I^-\Phi_J^-$$

where $\mathcal{H}$ is the Hamiltonian (2.13). From the above form it is clear that since the elements T_{ab} form a closed algebra of constraints, they will also form a closed algebra on the constraint surface $\Phi_I^- = 0$, since Φ_I^- appears quadratically in both T_{00} and T_{01}. The same applies if we switch from Poisson brackets to Dirac brackets.

We use the above results to calculate the energy-momentum tensor for the model we have been dealing with, where $j_\mu = A_{In} = \tilde{A}_{In} = B_{mn} = 0$. We set $\Phi_I^- = 0$ and denote $\Phi_I^+ \equiv \Phi_I$. In terms of the coordinates $\sigma^\pm$, the only non-zero components of T are $T_{\pm\pm} = \frac{1}{2}(T_{00} \pm T_{01})$, given explicitly by

$$T_{\pm\pm} = \frac{1}{8}\left(H^{IJ} \pm L^{IJ}\right)\Phi_I\Phi_J + \partial_\pm Y\partial_\pm Y + \eta_{ab}\partial_\pm Y^a\partial_\pm Y^b$$

where $\partial_\pm = \frac{1}{2}(\partial_0 \pm \partial_1)$. We now substitute in our mode expansions for Φ_I, Y, Y^m to obtain Virasoro operators $L_m, \tilde{L}_m$. We will do this for both sectors, to obtain physical state conditions for twisted and untwisted states. We begin with sector I.

Substituting in the untwisted expansions for Φ_I, Y, Y^a into the above gives the following expressions for the Virasoro operators:

$$\begin{aligned} L_m &= \frac{1}{2\pi}\int_0^{2\pi} e^{im\sigma^-}T_{--}d\sigma^- \\ &= \frac{1}{2}\sum_{k=-\infty}^{+\infty}(a_{m-k}a_k + b_{m-k}b_k + \eta_{ab}b^a_{m-k}b^b_k) \end{aligned}$$

where

$$a_0 \equiv \frac{1}{2}\left(\frac{q_1}{R} - q_2 R\right), \qquad b_0 \equiv \sqrt{2}p_R, \qquad b_0^a \equiv \frac{p^a}{\sqrt{2}} \tag{3.13}$$

Similarly,

$$\begin{aligned} \tilde{L}_m &= \frac{1}{2\pi}\int_0^{2\pi} e^{im\sigma^+}T_{++}d\sigma^+ \\ &= \frac{1}{2}\sum_{k=-\infty}^{+\infty}\left(\tilde{a}_{m-k}\tilde{a}_k + \tilde{b}_{m-k}\tilde{b}_k + \eta_{ab}\tilde{b}^a_{m-k}\tilde{b}^b_k\right) \end{aligned}$$

where

$$\tilde{a}_0 \equiv \frac{1}{2}\left(\frac{q_1}{R} + q_2 R\right), \qquad \tilde{b}_0 \equiv \sqrt{2}p_L, \qquad \tilde{b}_0^a \equiv \frac{p^a}{\sqrt{2}} \tag{3.14}$$

For the normal ordered zero modes L_0 and $\tilde{L}_0$ we have

$$\begin{aligned} L_0 &= \frac{1}{8}\left(\frac{q_1}{R} - q_2 R\right)^2 + p_R^2 + \frac{1}{4}(p^a)^2 + \sum_{k=1}^{\infty}(a_{-k}a_k + b_{-k}b_k + b^a_{-k}b^a_k) \\ \tilde{L}_0 &= \frac{1}{8}\left(\frac{q_1}{R} + q_2 R\right)^2 + p_L^2 + \frac{1}{4}(p^a)^2 + \sum_{k=1}^{\infty}(\tilde{a}_{-k}\tilde{a}_k + \tilde{b}_{-k}\tilde{b}_k + \tilde{b}^a_{-k}\tilde{b}^a_k) \end{aligned}$$

Therefore, the level matching condition is

$$\frac{1}{2}q_1 q_2 + p_L^2 - p_R^2 + \tilde{N} - N = 0 \tag{3.15}$$

Note that the first term will be an integer because we have the quantization condition $q_1 q_2 = 2mn$, $m, n \in \mathbb{Z}$. The mass spectrum formula is

$$M^2 = 2\left(p_L^2 + p_R^2 + \frac{q_1^2}{4R^2} + \frac{q_2^2 R^2}{4} + N + \tilde{N} - 2\right) \tag{3.16}$$

where the -2 arises as the zero point energy of 24 left-handed and 24 right-handed integer moded bosonic oscillators, which each contribute $-1/24$.

From the mass formula we see that the state $a_{-1}\tilde{a}_{-1}|k^a>$, which corresponds to the metric component along the fibre, is indeed massless, as one would expect. However, it belongs to $H^-_{(+)}$, i.e. it has eigenvalue -1 under the orbifold action. Therefore, this state will be projected out. This is in agreement with refs. [4] and [11], where it is explained that when there is a non-trivial monodromy the moduli must take values which are fixed under the action of the monodromy. In our example $R \to R^{-1}$, so the component of the metric with both legs in the fibre has fixed value 1. In orbifold language this means the corresponding state, $a_{-1}\tilde{a}_{-1}|k^a>$, must be projected out, which is indeed what we find here.

We now consider the energy-momentum tensor and physical state conditions for the twisted sector II. First note that

$$T_{--} = \frac{1}{8}\left(\frac{q}{R} - qR\right)^2 + \frac{1}{2}\left(\frac{q}{R} - qR\right)\sum_{k\in\mathbb{Z}+\frac{1}{2}} a_k e^{-ik\sigma^-} + \dots$$

That is, T_{--} has both integer and half integer modes; therefore it will be neither periodic nor antiperiodic. T_{++} is periodic and we require T_{--} to be periodic. This is only satisfied if $R = 1$. We put $R = 1$ from now on. We then obtain the following L_m and $\tilde{L}_m$,

$$\begin{aligned} L_m &= \frac{1}{2\pi}\int_0^{2\pi} e^{im\sigma^-} T_{--} d\sigma^- \\ &= \frac{1}{2}\sum_{k\in\mathbb{Z}+\frac{1}{2}} a_{m-k}a_k + \frac{1}{2}\sum_{k\in\mathbb{Z}}\left(b_{m-k}b_k + \eta_{ab} b^a_{m-k} b^b_k\right) \end{aligned}$$

where b_0 and b_0^a are related to the Y-momenta via (3.13). Similarly,

$$\begin{aligned} \tilde{L}_m &= \frac{1}{2\pi}\int_0^{2\pi} e^{im\sigma^+} T_{++} d\sigma^+ \\ &= \frac{1}{2}\sum_{k=-\infty}^{+\infty}\left(\tilde{a}_{m-k}\tilde{a}_k + \tilde{b}_{m-k}\tilde{b}_k + \eta_{ab}\tilde{b}^a_{m-k}\tilde{b}^b_k\right) \end{aligned}$$

where $\tilde{a}_0 = q$, and $\tilde{b}_0, \tilde{b}_0^a$ are related to the Y-momenta via (3.14). For the normal ordered zero modes, L_0 and $\tilde{L}_0$, we have

$$L_0 = p_R^2 + \frac{1}{4}(p^a)^2 + \sum_{k=\frac{1}{2}}^{\infty} a_{-k}a_k + \sum_{k=1}^{\infty}(b_{-k}b_k + b^a_{-k}b^a_k)$$

$$\tilde{L}_0 = \frac{1}{2}q^2 + p_L^2 + \frac{1}{4}(p^a)^2 + \sum_{k=1}^{\infty}(\tilde{a}_{-k}\tilde{a}_k + \tilde{b}_{-k}\tilde{b}_k + \tilde{b}^a_{-k}\tilde{b}^a_k)$$

The zero point energy for the right-movers will be -1, since we have a contribution of $-1/24$ from each of the 24 periodic bosons. On the left hand side the zero point energy is $-45/48$ since we have 23 periodic bosons contributing $-1/24$ and 1 anti-periodic boson contributing $+1/48$. So the condition on physical states is

$$(\tilde{L}_0 - 1)|phys>= (L_0 - \frac{45}{48})|phys>= 0$$

Hence the level matching condition and mass spectrum formula are given by

$$\frac{1}{2}q^2 + p_L^2 - p_R^2 + \tilde{N} - N - \frac{1}{16} = 0 \tag{3.17}$$

$$M^2 = 2\left(p_L^2 + p_R^2 + \frac{1}{2}q^2 + N + \tilde{N} - (2 - \frac{1}{16})\right) \tag{3.18}$$

The term $-1/16$ in the level matching condition looks problematic if the formula is written in terms of the original zero mode q. Level matching problems are well known to plague asymmetric orbifolds, and generally one must make some kind of fix to make the level matching formula sensible. The simplest solution here is to quantize q appropriately so that the factor of $-1/16$ cancels. This happens if we choose $\sqrt{2}q = n - 1/2$, $n \in \mathbb{Z}$ [15]. Moreover, in appendix B we show that this quantization rule follows directly from having the correct phase (3.11) for the action of T-duality. We now move on to investigate the partition function for this model. We will see that this quantization for q leads to a modular invariant partition function.

3.6 The partition function

We now have all the ingredients required to calculate the partition function. We are particularly interested in the partition function for the non-trivial part of the background, namely the fibre bundle over S^1. Following Flournoy and Williams [11] for the construction of partition functions for interpolating orbifolds, this should be given by

$$Z(\tau) = \frac{1}{2}\sum_{a,b=0,1} Z^a_{(\Phi)b}(\tau) Z^a_{(Y)b}(\tau) \tag{3.19}$$

where $Z^a{}_b$ is the partition trace associated to b insertions, with the trace taken over the Hilbert space H_a, i.e.

$$Z^a{}_b = \mathrm{Tr}_{H_a}(g^b q^{L_0} q^{\tilde{L}_0})$$

Here g is the orbifold action and $q = \exp(2\pi i\tau)$ as usual. In terms of our previous notation $H_0 \equiv H_{(+)}$ and $H_1 \equiv H_{(-)}$. So the essential point is that we are multiplying partition traces for the Φ and Y excitations together, rather than calculating the full Φ and Y partition functions separately and then multiplying the results. This is because we are dealing with an interpolating orbifold, rather than a simple product orbifold.

For the Φ excitations we obtain the following partition traces from the Hilbert spaces $H_{(\pm)}$ and $L_0, \tilde{L}_0$ found previously. For sector I we obtain

$$Z^0{}_0 = \frac{1}{|\eta|^2\sqrt{\tau_2}\epsilon} \sum_{m,n\in\mathbb{Z}} \exp\left(-\frac{\pi}{\tau_2\epsilon^2}|m+n\tau|^2\right) \tag{3.20}$$

$$Z^0{}_1 = \left(\frac{2\eta}{\theta_2}\right)^{1/2} \frac{\overline{\theta_4}(2\tau)}{\overline{\eta}} \tag{3.21}$$

where in both cases we have used the quantization rule $q_1 q_2 = 2mn$, $m, n \in \mathbb{Z}$, which implies

$$q_1 = \sqrt{2}m\epsilon, \qquad q_2 = \frac{\sqrt{2}n}{\epsilon}$$

for some $\epsilon \in \mathbb{R}$. For $Z^0{}_1$ the only states which contribute are those with $q_1 = q_2$, which implies $\epsilon = 1$ and $m = n$. For sector II we obtain

$$Z^1{}_0 = \left(\frac{\eta}{\theta_4}\right)^{1/2} \frac{\overline{\theta_2}(\frac{1}{2}\tau)}{\overline{\eta}} \tag{3.22}$$

$$Z^1{}_1 = \left(\frac{2\eta}{\theta_3}\right)^{1/2} \frac{\overline{\theta_2}(\frac{\tau}{2}; -\frac{1}{4})}{\overline{\eta}} \tag{3.23}$$

For completeness we give the partition traces for the Y excitations. These have been given in the following compact form in ref. [11],

$$Z^a_{(Y)b} = \sum_{n,w\in\mathbb{Z}} \sum_{q=0,1} (-1)^{bq} Z_{2R}\left[2n+q \middle| w + \frac{a}{2}\right]$$

where the definition of $Z_{2R}[\ldots|\ldots]$ can be found in the appendix C.

The partition traces for both the fibre and base directions are modular covariant, which implies the full partition function (3.19) is modular invariant. The modular covariance properties are

$$Z(\tau+1)^a{}_b = Z(\tau)^a{}_{b-a}, \qquad Z(-1/\tau)^a{}_b = Z(\tau)^b{}_{-a}$$

for each a, b. To see these conditions are satisfied one must use some subtle properties of the θ functions, which are summarised in appendix C. The modular covariance of the Φ partition traces relies both on the quantization condition for the zero modes and the phase factors, both of which were introduced in ref. [15]. This improves on earlier work [11, 23] where this orbifold was not found to be modular covariant. This is due to not having the "correct" realization of T-duality on the different sectors.

So we have shown that the doubled S^1 system, considered as a constrained Hamiltonian system, is equivalent quantum mechanically to the conventional non-doubled picture. That is, one obtains the same partition function as obtained in ref. [15]. An important point is that we have not needed to make any choice of physical states. Even though we haven't chosen a polarization it is not surprising that we obtain the same partition function. This is because T-dual theories have the same partition function.

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4. The supersymmetric doubled torus

An obvious extension to the doubled torus formalism is to make the Lagrangian and the associated constraint supersymmetric. This will allow more complicated orbifolds (hopefully modular invariant, and perhaps realistic) to be considered from the doubled torus perspective. We have completed the first step, which is simply to find the supersymmetric doubled torus Lagrangian and the relevant constraints. However, we leave the problem of constructing supersymmetric orbifolds from this perspective to future work. Note that supersymmetric asymmetric orbifolds corresponding to T-folds have been considered in [11, 15], but not from the doubled formalism/constrained Hamiltonian point of view.

4.1 Extending the lagrangian

We want to make the doubled torus Lagrangian (2.2) and the constraints (2.4) supersymmetric. We use the following definitions for superfields, which are supersymmetric extensions of our X, Y:

$$\mathbb{X}^I = X^I + \bar{\theta}\psi^I + \frac{1}{2}(\bar{\theta}\theta)F^I$$

$$\mathbb{Y}^n = Y^n + \bar{\theta}\chi^n + \frac{1}{2}(\bar{\theta}\theta)\phi^n$$

or collectively

$$\mathcal{Q}^\mu = q^\mu + \bar{\theta}\psi^\mu + \frac{1}{2}(\bar{\theta}\theta)f^\mu$$

Covariant derivatives are defined as follows

$$D_\alpha\mathcal{Q}^\mu = \psi^\mu_\alpha + \theta_\alpha f^\mu - i(\rho^a\theta)_\alpha\partial_a q^\mu + \frac{i}{2}\partial_a(\rho^a\psi^\mu)_\alpha(\bar{\theta}\theta)$$

Our conventions are given in appendix A. We study the following Lagrangian:

$$\mathcal{L} = \int d^2\theta \left\{\frac{1}{2}g_{\mu\nu}(\mathbb{Y})\bar{D}\mathcal{Q}^\mu D\mathcal{Q}^\nu - \frac{1}{2}b_{\mu\nu}(\mathbb{Y})\bar{D}\mathcal{Q}^\mu(\rho_3)D\mathcal{Q}^\nu\right\} \tag{4.1}$$

$$= \int d^2\theta\Big\{\frac{1}{2}H_{IJ}(\mathbb{Y})\bar{D}\mathbb{X}^I D\mathbb{X}^J + A_{Im}(\mathbb{Y})\bar{D}\mathbb{X}^I D\mathbb{Y}^m - \tilde{A}_{Im}(\mathbb{Y})\bar{D}\mathbb{X}^I(\rho_3)D\mathbb{Y}^m$$
$$+\frac{1}{2}G_{mn}(\mathbb{Y})\bar{D}\mathbb{Y}^m D\mathbb{Y}^n - \frac{1}{2}B_{mn}(\mathbb{Y})\bar{D}\mathbb{Y}^m(\rho_3)D\mathbb{Y}^n\Big\} \tag{4.2}$$

where $\rho_3 = \rho^0\rho^1 = \sigma^3$, the third Pauli matrix[4] and $b_{\mu\nu}$ has non-zero components $b_{Im} = -b_{mI} = \tilde{A}_{Im}$ and $b_{mn} = B_{mn}$. Note that all the spinor indices in the above equations are contracted. We integrate just the fermionic part, using $\int d^2\theta(\bar{\theta}\theta) = 1$, to obtain a supersymmetric Lagrangian. This gives the correct bosonic Lagrangian upon truncation, i.e. we obtain the original bosonic doubled torus Lagrangian (2.2).

In more detail, we expand each superfield term in its constituents. For example, the first term in (4.2) is expanded as follows,

$$H_{IJ}(\mathbb{Y})\bar{D}_\alpha\mathbb{X}^I D_\alpha\mathbb{X}^J = H_{IJ}(\mathbb{Y})\bar{\psi}^I\psi^J + 2H_{IJ}(\mathbb{Y})\bar{\psi}^I\theta F^J$$

[4]Interestingly, $\rho^3 \cdot V = -\star V$, where $V = V_a\rho^a$, that is the volume element acts like the Hodge dual.

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$$
\begin{aligned}
&- 2iH_{IJ}(\mathbb{Y})(\bar{\psi}^I\rho^a\theta)\partial_a X^J \\
&+ H_{IJ}(\mathbb{Y})\left(\eta^{ab}\partial_a X^I\partial_b X^J + i\bar{\psi}^I\rho^a\partial_a\psi^J + F^I F^J\right)\bar{\theta}\theta
\end{aligned}
$$

where

$$
\begin{aligned}
H_{IJ}(\mathbb{Y}) = H_{IJ}(Y) + \partial_n H_{IJ}(Y)\bar{\theta}\chi^n + &\frac{1}{2}\partial_n H_{IJ}(Y)\bar{\theta}\theta\phi^n \\
&+ \frac{1}{2}\partial_m\partial_n H_{IJ}(Y)(\bar{\theta}\chi^m)(\bar{\theta}\chi^n)
\end{aligned}
$$

The only terms that contribute to the Lagrangian are those which are coefficients of $\bar{\theta}\theta$ in the expansion. Expanding everything in this way and integrating we arrive at the following supersymmetric Lagrangian,

$$
\begin{aligned}
\mathcal{L} = &\frac{1}{2}g_{\mu\nu}\partial_a q^\mu\partial_b q^\nu\eta^{ab} - \frac{1}{2}b_{\mu\nu}\partial_a q^\mu\partial_b q^\nu\epsilon^{ab} \\
&+\frac{1}{2}g_{\mu\nu}i\bar{\psi}^\mu \not{\partial}\psi^\nu - \frac{i}{2}b_{\mu\nu}\bar{\psi}^\mu\rho^3 \not{\partial}\psi^\nu \\
&+\frac{1}{2}g_{\rho\sigma,\nu}i\bar{\psi}^\rho\rho^a\psi^\nu\partial_a q^\sigma - \frac{1}{2}b_{\nu\rho,\mu}i\bar{\psi}^\nu\rho^3\rho^a\psi^\mu\partial_a q^\rho \\
&+\frac{1}{2}g_{\mu\nu}f^\mu f^\nu + \left(-\frac{1}{2}\Gamma^\mu_{\rho\nu}\bar{\psi}^\rho\psi^\nu - \frac{1}{4}H^\mu{}_{\rho\nu}\bar{\psi}^\rho\rho^3\psi^\nu\right)g_{\mu\kappa}f^\kappa \\
&-\frac{1}{8}g_{\rho\sigma,\mu\nu}\bar{\psi}^\mu\psi^\nu\bar{\psi}^\rho\psi^\sigma + \frac{1}{8}b_{\rho\sigma,\mu\nu}\bar{\psi}^\rho\rho^3\psi^\sigma\bar{\psi}^\mu\psi^\nu
\end{aligned} \tag{4.3}
$$

Substituting for f^μ and after some algebra we obtain the following Lagrangian with auxiliary fields solved,

$$
\begin{aligned}
\mathcal{L} = &\frac{1}{2}g_{\mu\nu}\partial_a q^\mu\partial_b q^\nu\eta^{ab} - \frac{1}{2}b_{\mu\nu}\partial_a q^\mu\partial_b q^\nu\epsilon^{ab} \\
&+\frac{1}{2}g_{\mu\nu}i\bar{\psi}^\mu \not{\nabla}^+\psi^\nu + \frac{1}{4}R^-_{\mu\nu\rho\sigma}\psi^\mu_+\psi^\nu_+\psi^\rho_-\psi^\sigma_-
\end{aligned} \tag{4.4}
$$

where $\nabla^\pm_\mu V^\nu = \nabla_\mu V^\nu \mp \frac{1}{2}H_\mu{}^\nu{}_\rho V^\rho$ and $R^{-\mu}{}_{\nu\rho\sigma} = \left[\nabla^-_\rho, \nabla^-_\sigma\right]^\mu{}_\nu$. The operator $\not{\nabla}^\pm = \partial_a q^\mu\rho^a\nabla^\pm_\mu$, i.e. the pull-back of $\nabla^\pm$ to the world-sheet. One could, of course, now expand the above Lagrangian in terms of the original data H_{IJ}, A_{Im}, $\tilde{A}_{Im}$, B_{mn}. We will not do this here as the expanded form will not be needed in the following.

4.2 Supersymmetric constraints

We now turn to the constraint. The constraint of the bosonic theory (2.4) can be written equivalently as

$$
\dot{X} - L\tilde{A}\dot{Y} = S(X' - L\tilde{A}Y') \tag{4.5}
$$

an equation that halves the independent vectors $\{dX^I\} \in X^\star T(T^{2n})$ on the doubled torus, where $X^\star$ is the pull-back of the map $X : \Sigma \to T^{2n}$. The fermions in the supersymmetric sigma model are sections of the $X^\star T(T^{2n}) \otimes \sqrt{K}$ bundle and it is natural to halve the independence of them too. Furthermore, the constraints obtained should be supersymmetric. We find the following constraint sufficient,

$$
D_\alpha\mathbb{X}^I - L^{IJ}\tilde{A}_{Jn}(\mathbb{Y})D_\alpha\mathbb{Y}^n = -S^I{}_J(\mathbb{Y})\rho^3_{\alpha\beta}\left(D_\beta\mathbb{X}^J - L^{JK}\tilde{A}_{Kn}(\mathbb{Y})D_\beta\mathbb{Y}^n\right) \tag{4.6}
$$

We also require the same consistency condition (2.5) in its functional form unchanged, i.e.

$$A_{In}(\mathbb{Y}) = -H_{IJ}(\mathbb{Y})L^{JK}\tilde{A}_{Kn}(\mathbb{Y})$$

The constraint in (4.6) reduces to (4.5) upon setting fermions and auxiliary fields to zero.

Now we consider the constraint (4.6) with all fields turned on, at each order in θ. Firstly, the constant term reads

$$\psi^I - L^{IJ}\tilde{A}_{Jn}(Y)\chi^n = -S^I{}_J(Y)\rho^3\psi^J + H^{IJ}\tilde{A}_{Jn}\rho^3\chi^n \tag{4.7}$$

This halves independence of the fermions ψ^I using an endomorphism of the target tangent vector bundle. A nice way of writing this is to split the fermions in their chiral parts. Then the above constraint becomes

$$\begin{aligned}(1+S)\psi_+ &= (1+S)L\tilde{A}\chi_+\\ (1-S)\psi_- &= (1-S)L\tilde{A}\chi_-\end{aligned} \tag{4.8}$$

These constraints seem very natural as $\frac{1}{2}(1 \pm S)$ are projectors. Therefore, half of the ψ^Is are constrained to be given in terms of the χ^ms. From the linear terms in θ we obtain the following constraints:

$$\begin{aligned}\dot{X} - SX' - L\tilde{A}\dot{Y} + H^{-1}\tilde{A}Y' &= -\frac{i}{2}S\bar{\chi}^n\rho^1\partial_n\left(L\tilde{A}\chi - S\rho^3\psi + H^{-1}\tilde{A}\rho^3\chi\right)\\ f - L\tilde{A}\phi &= -\frac{1}{2}\bar{\chi}^n\partial_n\left(L\tilde{A}\chi - S\rho^3\psi + H^{-1}\tilde{A}\rho^3\chi\right)\end{aligned}$$

The first equation is clearly the initial bosonic constraint (4.5) on the left hand side, generalized by the addition of some fermionic terms on the right hand side. In phase space it can be written in exactly the same way as the original constraint, namely

$$\pi_I - L_{IJ}X'^J = 0$$

where π_I is the canonical momentum associated to X^I derived from the supersymmetric Lagrangian (4.4). The second equation above is automatically satisfied when the auxiliary fields are put on-shell.

We now turn to the quadratic θ term of the constraint. In particular, we show how this is automatically satisfied if the constant and linear terms are imposed and conserved on shell (i.e. the time derivatives of these constraints are also satisfied). First note that we can collect the equation of motion for $\mathbb{X}^I$ in supersymmetric form from (4.1) as

$$\bar{D}_\alpha\left(g_{I\mu}(\mathbb{Y})D_\alpha\mathcal{Q}^\mu - b_{I\mu}(\mathbb{Y})\rho^3_{\alpha\beta}D_\beta\mathcal{Q}^\mu\right) = 0 \tag{4.9}$$

or

$$\bar{D}_\alpha\left(H_{IJ}(\mathbb{Y})D_\alpha\mathbb{X}^J + A_{In}(\mathbb{Y})D_\alpha\mathbb{Y}^n - \tilde{A}_{In}(\mathbb{Y})\rho^3_{\alpha\beta}D_\beta\mathbb{Y}^n\right) = 0$$

Similarly the constraint (4.6) can be written as

$$H_{IJ}(\mathbb{Y})D_\alpha\mathbb{X}^J + A_{In}(\mathbb{Y})D_\alpha\mathbb{Y}^n - \tilde{A}_{In}(\mathbb{Y})\rho^3_{\alpha\beta}D_\beta\mathbb{Y}^n + L_{IJ}\rho^3_{\alpha\beta}D_\beta\mathbb{X}^J = C_{I\alpha} = 0 \tag{4.10}$$

Note that $\bar{D}_\alpha \rho^3_{\alpha\beta} D_\beta = 0$ as a consequence of the supersymmetry algebra. Therefore, the constraint implies the equations of motion for $\mathbb{X}$, in complete analogy with the bosonic constraint implying the equation of motion for X^I [1]. That is, schematically we have

$$C^I_\alpha = 0 \Rightarrow \bar{D}_\alpha C^I_\alpha = 0 \Leftrightarrow \mathrm{eom}(\mathbb{X})$$

By writing the constraint expansion as

$$C^I_\alpha = C^{I(0)}_\alpha + \bar{\theta}_\beta C^{I(1)}_{\alpha\beta} + \frac{1}{2}(\bar{\theta}\theta) C^{I(2)}_\alpha$$

we can show how

$$\left.\begin{aligned} C^{I(0)}_\alpha &= 0 \quad \text{on shell} \\ C^{I(1)}_{\alpha\beta} &= 0 \quad \text{on shell} \\ \bar{D}_\alpha C^I_\alpha &= 0 \ \text{eom for } \mathbb{X}^I \end{aligned}\right. \Longrightarrow C^{I(2)}_\alpha = 0$$

Thus our supersymmetric constraint (4.6) makes sense. That is, it halves the fermionic and bosonic degrees of freedom, without imposing extra unphysical constraints. The two constraints arising from (4.6) are thus

$$\pi_I - L_{IJ} X'^J = 0 \tag{4.11}$$

$$(1 + \rho^3 S)_I{}^J g_{J\mu} \psi^\mu = 0 \tag{4.12}$$

where the first equation is our original bosonic constraint plus corrections.

5. Conclusion

In this paper we have shown that by applying methods from constrained Hamiltonian systems one finds that the doubled torus system is equivalent quantum mechanically to the non-doubled system, at least for the simple example we have worked out here. Previously, this equivalence was only established classically, and these methods had not been applied.

The doubled torus system proposed by Hull [1] is a constrained Lagrangian system, and the natural formalism for understanding these systems is the methods of constrained Hamiltonian systems, where the dynamics is considered on the constrained surface. Therefore, our work is the natural extension of ref. [1] where the Lagrangian formalism was used. By moving to phase space, and defining a Poisson structure, we find that we do not need to choose a polarization for our new variables Φ^+, and we construct a polarization invariant Hilbert space. Making use of the results of ref. [15] for the action of T-duality on states, we find that our Hilbert space leads to a modular invariant partition function, which is exactly the same as that of the non-doubled theory. This is not surprising since T-dual theories should have the same Hilbert space and partition fucntions, and the doubled torus is, in some sense, the set of all T-duals of a given T-fold.

Note that although we have not needed to choose a polarization, if we wanted to interpret our constrained Hamiltonian as a sigma model without constraints, this would

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involve choosing a polarization for Φ^+. In particular, one would need to choose which of the Φ^+ variables are the momenta.

The zero mode quantization is very interesting. In particular, we show that knowing the phase [15] in the action of T-duality leads to the correct zero mode quantization. Our construction for proving this quantization is an orbifold one, as opposed to a more general Wilson line theory such as those proposed in ref. [15].

The final part of our paper deals with constructing a consistent supersymmetric extension to the doubled torus formalism. This involves making the constraint supersymmetric, and then checking that the superfield constraint does not impose too many restrictions on the constituent fields, which would be unphysical. Surprisingly, the constraints turn out to be very simple, both in the superfield language, and when expanded out as coefficients of θ. Our final result is that we have n bosonic constraints, which contain the original constraint plus fermionic corrections, and n new fermionic constraints.

The doubled torus system is a tractable example of a constrained Hamiltonian system because its Dirac brackets are very simple, allowing us to implement Dirac bracket quantization, at least for the simple flat background we have considered. For curved backgrounds this is generally not possible and one must use a more complicated method of quantization, such as that used in ref. [29]. In the supersymmetric case we find that everything is very similar to the bosonic case, and all of the constraints are second class. It would be interesting to investigate the quantization of the supersymmetric doubled torus and consider associated asymmetric orbifolds. Note that although we have only considered a very simple example of a T-fold, we expect other examples to follow through in the same vein, and to also display quantum mechanical equivalence between the doubled and non-doubled formulations.

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Acknowledgments

We would like to thank James Gray and Harry Braden for useful comments on a draft version of this paper. We are extremely grateful to José Figueroa-O'Farrill for suggesting this project and for many very helpful discussions. E. H-J. is supported by an EPSRC Postdoctoral Fellowship.

A. Conventions

Our worldsheet metric has signature $(+,-)$. For the Clifford algebra we define $\{\rho^a, \rho^b\} = 2\eta^{ab}$, where η is the flat metric. Whenever needed we will use the representation

$$\rho^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\rho^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

In $1+1$ dimensions one has the choice of Dirac, Majorana, Weyl or Majorana-Weyl spinors. We choose to work with real Majorana spinors.

Since we are considering $\mathcal{N}=1$ supersymmetry on the worldsheet, our superfields will involve one Majorana spinor parameter θ_α, Grassmann odd in nature. The supercharges are defined as follows

$$Q_\alpha = \frac{\partial}{\partial \bar\theta^\alpha} + i(\rho^a\theta)_\alpha \partial_a$$

$$\bar Q_\alpha = (Q^* \rho^0)_\alpha = -\frac{\partial}{\partial \theta^\alpha} - i(\bar\theta\rho^a)_\alpha \partial_a$$

$$\{Q_\alpha, Q_\beta\} = -2i(\rho^a\rho^0)_{\alpha\beta}\partial_a$$

where $\bar\theta_\alpha = \theta_\beta \rho^0_{\beta\alpha}$ as usual. We introduce the super-derivatives

$$D_\alpha = \frac{\partial}{\partial \bar\theta^\alpha} - i(\rho^a\theta)_\alpha \partial_a$$

$$\bar D_\alpha = (D^* \rho^0)_\alpha = -\frac{\partial}{\partial \theta^\alpha} + i(\bar\theta\rho^a)_\alpha \partial_a$$

$$\{D_\alpha, D_\beta\} = 2i(\rho^a\rho^0)_{\alpha\beta}\partial_a$$

For these we use the fact that $\overline{(\rho^\alpha\theta)}_\alpha = (\bar\theta\rho^a)_\alpha$ and $\overline{(\frac{\partial}{\partial\theta^\alpha})} = -\frac{\partial}{\partial\theta^\alpha}$. Note that the super-derivatives anti-commute with the charges,

$$\{Q_\alpha, D_\beta\} = 0$$

Our superfields $\mathbb{X}$, $\mathbb{Y}$ are supersymmetric extensions of our X, Y:

$$\mathbb{X}^I = X^I + \bar\theta\psi^I + \frac{1}{2}\bar\theta\theta f^I$$

$$\mathbb{Y}^n = Y^n + \bar\theta\chi^n + \frac{1}{2}\bar\theta\theta \phi^n$$

or collectively

$$\mathcal{Q}^\mu = q^\mu + \bar\theta\psi^\mu + \frac{1}{2}\bar\theta\theta f^\mu$$

The covariant derivative of $\mathbb{X}$ is given by

$$D_\alpha \mathbb{X}^I = \Psi^I_\alpha + \theta_\alpha F^I - i(\rho^a\theta)_\alpha \partial_\alpha X^I + \frac{i}{2}\partial_a(\rho^a\psi^I)_\alpha \bar\theta\theta$$

where we have used the Fierz identity $\theta_\alpha\bar\theta_\beta = -\frac{1}{2}\delta_{\alpha\beta}\bar\theta\theta$, which implies the useful relation $\bar\theta\epsilon_1\bar\theta\epsilon_2 = -\frac{1}{2}\bar\epsilon_2\epsilon_1\bar\theta\theta$.

B. Quantization of the zero modes

In this section we describe how to obtain the quantization of the zero modes of Φ_I.

First, let's recall the simple case of a quantum point particle on a circle S^1, considered as an orbifold $\mathbb{R}/\mathbb{Z}$. The Hilbert space on $\mathbb{R}$ is made up of momentum states $|p>_{p\in\mathbb{R}}$. Calling the generator of translations $t: x \to x + 2\pi$, we have that $t|p>= \exp(i2\pi p)|p>$. The invariant Hilbert space consists of the projected states

$$\sum_n t^n|p> = \sum_n \exp(in2\pi p)|p> = \delta(2\pi p)|p>$$

which implies that $p = 0 \mod 1$. If for some reason the momentum was initially quantized in even integers, on the circle the momentum can be further fractionated to take any integer value. Furthermore, we want the operator $\exp(ix)$ to be realised on the Hilbert space and this will require all integer values of momentum to be taken into account.

Now let's turn to sector I of our model. The constraint (2.4) halves the physical degrees of momentum, winding and oscillator modes. Because it is a differential constraint, the number of zero modes of X^I will not be halved. Therefore, we must put in an extra constraint on X_0^I, so that we have the correct number of degrees of freedom of a string theory. The natural constraint to implement is

$$\Pi^i{}_I X_0^I = X_0^i, \qquad \tilde{\Pi}_{\underline{i}I} X_0^I = 0$$

where $\Pi^i{}_I$ and $\tilde{\Pi}_{\underline{i}I}$ are the projectors discussed in our section 2. The indices i correspond to the physical polarization, and $\underline{i}$ to the unphysical polarization. From section 2.2 we have that X and Φ obey the following Dirac bracket,

$$\left\{X^I(\sigma), \Phi_J(\sigma')\right\}_{\mathrm{D}} = \delta^I_J \delta(\sigma - \sigma')$$

Hence, once we quantize, we can extract the following commutator

$$[X_0^i, \Phi_{0j}] = \delta^i_j$$

where Φ_{0j} is the "physical" component. Therefore, Φ_{0i} can be thought of as the conjugate momenta to X_0^i. Hence $\Phi_{0i} \in \mathbb{Z}$, just as in the case of a quantum point particle on a circle.

For the other polarization, $\Phi_{0\underline{i}}$, we can use the fact that $L^{IJ}\Phi_J \sim 2X'^I$ (up to additions of Φ^- which we have set to zero). Therefore $\Phi_{0\underline{i}}$ obtains the quantization from the winding modes, and we have $\Phi_{0\underline{i}} \in 2\mathbb{Z}$.

In matrix form, these conditions can be written concisely as

$$\Pi\Phi_0 = m, \qquad \tilde{\Pi}\Phi_0 = 2n$$

where we are now thinking of Φ as a column vector, and $m, n \in \mathbb{Z}$. Then using the relation

$$(\Pi)^T\tilde{\Pi} + (\tilde{\Pi})^T\Pi = L$$

we arrive at the covariant quantization condition

$$\Phi_0^T L \Phi_0 = 4mn \tag{B.1}$$

We will now show an alternative derivation of this quantization for Φ_0 in sector I of our $T^2 \times \mathbb{R} \times N/\mathbb{Z}$ (plus constraint) model. Then we will use the same method to derive the quantization rule in sector II. The generator of $\mathbb{Z}$ will be our orbifold transformation, g. The generator g acts like M on the fibre and translates by $2\pi R_y$ on the base circle. We write the zero modes as

$$\Phi_0 = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$$

We have the following action of g on the Hilbert space

$$g|q_1, q_2, n_y >= \exp\left(i\pi(n_y + \frac{q_1 q_2}{2})\right) |q_2, q_1, n_y >$$

The factor of $\exp(i\pi n_y)$ is the usual phase coming from the translation on the circle base. The phase $\exp(i\pi q_1 q_2/2)$ is known to be the right T-duality realisation for closure of OPEs in sector I (see eg. [15]). At this stage we don't restrict the quantization of q_1,q_2. After projection with $\sum_n g^n$, the existence of invariant states requires

$$\pi\left(\frac{q_1 q_2}{2} + n_y\right) = 0 \mod 2\pi$$

or for generic n_y:

$$q_1 q_2 = 2mn, \qquad m, n \in \mathbb{Z}$$

This is precisely the quantization condition (B.1).

We finally turn to sector II. We use the results of [15]. In their paper they solve issues like modular invariance and level matching for asymmetric orbifolds. Our case is what they call "tame" and our starting point is the phase of the T-duality. We write the zero mode as

$$\Phi_0 = \begin{pmatrix} q \\ q \end{pmatrix}$$

Our generator acts as

$$g|q, n_y >= \exp\left(i\pi(q^2 - \frac{1}{8} + n_y)\right) |q, n_y >$$

Our construction is an orbifold one and we can show modular invariance, level matching and quantization of zero modes by adopting the above phase. The invariant Hilbert space requires (for generic n_y):

$$q^2 - \frac{1}{8} = 0 \mod 1$$

The simplest choice with even spacing of the modes q is then

$$q = \frac{1}{\sqrt{2}}\left(n - \frac{1}{2}\right) \tag{B.2}$$

where $n \in \mathbb{Z}$.

C. Properties of θ functions

The θ functions we use are given by

$$\begin{aligned} \theta_2(\tau; z) &= \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}(n-\frac{1}{2})^2} e^{i\pi(2n-1)z} \\ \theta_3(\tau; z) &= \sum_{n\in\mathbb{Z}} q^{\frac{1}{2}n^2} e^{i2\pi n z} \end{aligned} \tag{C.1}$$

$$\theta_4(\tau;z) = \sum_{n\in\mathbb{Z}} (-1)^n q^{\frac{1}{2}n^2} e^{i2\pi nz} \tag{C.2}$$

where $q = \exp(2\pi i\tau)$ as usual. Usually we will take $z = 0$, and we denote $\theta_i(\tau;0) \equiv \theta_i$. The θ functions can also be written as infinite products as follows,

$$\begin{aligned}
\theta_2(\tau;z) &= 2\eta q^{\frac{1}{12}} \cos(\pi z) \prod_{n=1}^{\infty} (1 - 2q^n \cos(2\pi z) + q^{2n}) \\
\theta_3(\tau;z) &= \eta q^{-\frac{1}{24}} \prod_{n=1}^{\infty} \left(1 + 2q^{n-\frac{1}{2}} \cos(2\pi z) + q^{2n-1}\right) \\
\theta_4(\tau;z) &= \eta q^{-\frac{1}{24}} \prod_{n=1}^{\infty} \left(1 - 2q^{n-\frac{1}{2}} \cos(2\pi z) + q^{2n-1}\right)
\end{aligned} \tag{C.3}$$

where

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) \tag{C.4}$$

The following modular transformation properties will be useful,

$$\begin{aligned}
\eta(\tau+1) &= e^{\frac{i\pi}{12}} \eta(\tau) \\
\theta_2(\tau+1;z) &= e^{\frac{i\pi}{4}} \theta_2(\tau;z) \\
\theta_3(\tau+1;z) &= \theta_4(\tau;z) \\
\theta_4(\tau+1;z) &= \theta_3(\tau;z)
\end{aligned} \tag{C.5}$$

We also derive the identity

$$\sqrt{2}\theta_2\left(\tau + \frac{1}{2}; -\frac{1}{4}\right) = e^{i\frac{\pi}{8}} \theta_2(\tau;0) \tag{C.6}$$

by using the summation definition in (C.1) and pairing positive with negative modes appropriately. This is used to show $Z^1{}_0(\tau+1) = Z^1{}_1(\tau)$. We also have the following transformations,

$$\begin{aligned}
\eta\left(-\frac{1}{\tau}\right) &= (-i\tau)^{\frac{1}{2}} \eta(\tau) \\
\theta_2\left(-\frac{1}{\tau}; \frac{z}{\tau}\right) &= (-i\tau)^{\frac{1}{2}} e^{\frac{i\pi z^2}{\tau}} \theta_4(\tau;z) \\
\theta_3\left(-\frac{1}{\tau}; \frac{z}{\tau}\right) &= (-i\tau)^{\frac{1}{2}} e^{\frac{i\pi z^2}{\tau}} \theta_3(\tau;z) \\
\theta_4\left(-\frac{1}{\tau}; \frac{z}{\tau}\right) &= (-i\tau)^{\frac{1}{2}} e^{\frac{i\pi z^2}{\tau}} \theta_2(\tau;z)
\end{aligned} \tag{C.7}$$

These properties are what is required to show that (3.20)-(3.23) satisfy the correct modular covariance properties.

For the Y partition traces we need the following expression [11] for $Z_{2R}[\ldots|\ldots]$,

$$\begin{aligned}
&Z_{2R}\left[2n+q \middle| w + \frac{a}{2}\right] = \\
&\frac{1}{|\eta(\tau)|^2} \exp\left[-\pi\tau_2\left(\frac{(2n+q)^2}{4R^2} + 4R^2(w+\frac{a}{2})^2\right) + 2\pi i\tau_1(2n+q)(w+\frac{a}{2})\right]
\end{aligned} \tag{C.8}$$

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